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Fixed point results of α-β_Y-F-Geraghty type contractive mapping on modular b-metric spaces

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Abstract. In this paper, we generalize α-β_Y-F-Geraghty type contraction in modular b-metric spaces and prove some fixed point results that are justified by suitable examples. The obtained results improve and extend some well known fixed point results in the literature.

1. Introduction

The Banach fixed point theorem [3] is regarded as one of the most important result in the field of Functional Analysis. Subsequently, this theorem has been extended in numerous way, achieved either by relaxing the conditions imposed on the various generalized metric spaces or by incorporating the generalized contractive conditions. To solve an issue with measurable functions, Bakhtin [2] invented the concept of b-metric space in 1989. Czerwik [9] explicitly defined it and proved Banach contraction principle in this new metric space. However, it is shown in [17] that in general, a b-metric function $d(x, y)$ for $b > 1$ need not be continuous in both variables. Banach contaction theorem is also generalized in modular spaces which was introduced by Chistyakov [4–7] by developing the theory of this space for arbitrary non empty set. Further, he established that metric modular function is a generalized form of metric function. The distance function of metric space $d(p, q)$ has been replaced by $\omega_\lambda(p, q)$ for each $\lambda > 0$. In modular metric space, modular convergence, modular limit and modular completeness are "weaker" than the corresponding metric space. These characteristics of this space enhance the applicabilities of abstract spaces in many more research areas.

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In 2018, Ege and Alace [10] introduced the notion of modular b-metric space and established some fixed point results in it.

Rakotch [21] defined a new class of functions $\boldsymbol{\beta}$ in metric space by replacing the constant of Banach contraction principle [3] by the function β as follows:

Let (M, d) be a metric space and $\boldsymbol{\beta}$ be a family of functions denoted by β. Then for all $p, q \in M$, $\beta \in \boldsymbol{\beta}$ satisfies the following conditions:

(i) $\beta(p, q) = \beta(d(p, q))$, i.e., β is dependent on the distance between p and q only;
(ii) $0 \leq \beta(d(p, q)) < 1$ for every $d(p, q) > 0$;
(iii) β is a monotonically decreasing function, where $0 \leq \beta < 1$,

and established some fixed point results.

In [13] Geraghty replaced the Cauchy's condition for convergence of contractive iteration in a complete metric space by an equivalent functional condition. Huang et al. [16] generalized the Geraghty function in the setting of b-metric spaces. Samet et al. [22] introduced the concept of α-ψ-contractive and α-admissible mappings. Popescu [20] studied the existence and uniqueness of fixed point of generalized α-Geraghty contractive type mappings in complete metric spaces. Wardowski [23] introduced a new type of contraction called F-contraction. Cosentino et al. [8] generalized the Wardowski type contraction to solve certain classes of integrodifferential problems by introducing an additional condition for the class of auxiliary functions in complete b-metric spaces. Since then numourous authors have articulated the contractive conditions found in the literature in the context of F-contraction [14, 15, 18, 24].

In 2018, Lukács and Kajántó [18] generalize F-contraction defined by Wardowski [23]. In [23] Wardowski took three general conditions on function F and one extra contractive condition conserning the operator, while Lukács and Kajántó [18] proved fixed point theorem for different notions of F-contraction in b-metric space, without using Wardowski condition (F_2). Recently, Fulga and Proca [12] generalized F-contraction as F_Y-contraction in metric space and prove some fixed point theorems.

In 2019, Aydi et al. [1] introduced the notion of α-β_Y-Geraghty contractive type mappings in b-metric spaces. In this manuscript, we define (α, β_Y, F)-Geraghty type contraction in modular b-metric space and proved some fixed point results using it inspired by Aydi et al. [1].

2. PRELIMINARIES

We begin with the definition of b-metric space defined by Czerwik [9].

Definition 1 ([9])**.** Let M be a non-empty set , $b \geq 1$ be a real number. A function d, defined as $d : M \times M \to [0, \infty]$ is called a modular b-metric, if the following statements hold for all $p, q, r \in M$:

(1) $d(p,q) = 0 \Leftrightarrow p = q$,
(2) $d(p,q) = d(q,p)$,
(3) $d(p,r) \leq b[d(p,q) + d(q,r)]$.

Then, (M, d, b) is said to be a b-metric space.

Modular metric space was introduced by Chistyakov [7], who developed the theory of this space and derived some results in [4–6]. Consistent with [7], we begin with some basic definitions of modular metric spaces.

Let M be a non-empty set. Throughout this paper, for a function

$$\omega\colon (0,\infty) \times M \times M \to [0,\infty],$$

we write as $\omega(\lambda, p, q) \to \omega_\lambda(p,q)$, for all $p, q \in M_\omega$.

Definition 2 ([7])**.** Let M be a non-empty set, for a function $\omega\colon (0,\infty) \times M \times M \to [0,\infty]$ is said to be a metric modular on M if it satisfies, for all $p, q, r \in M$, the following conditions:

(i) $\omega_\lambda(p,q) = 0$ if and only if $p = q$ for all $\lambda > 0$;
(ii) $\omega_\lambda(p,q) = \omega_\lambda(q,p)$ for all $\lambda > 0$;
(iii) $\omega_{\lambda+\mu}(p,q) \leq \omega_\lambda(p,r) + \omega_\mu(r,q)$ for all $\lambda, \mu > 0$.

Then $(M_\omega, \omega_\lambda)$ is called modular metric space.

Definition 3 ([7])**.** Let M_ω be a modular metric space,$\{p_n\}_{n\in\mathbb{N}}$ be a sequence of M_ω. Then

(i) $\{p_n\}_{n\in\mathbb{N}}$ is said to be ω-convergent to $p \in M_\omega$ if and only if $p_n \to p$ if $\omega_\lambda(p_n, p) \to 0$ as $n \to \infty$ for all $\lambda > 0$.
(ii) $\{p_n\}_{n\in\mathbb{N}}$ is said to be ω-Cauchy sequence if $\omega_\lambda(p_m, p_n) \to 0$, as $m, n \to \infty$ for all $\lambda > 0$.
(iii) A subset C of M_ω is said to be complete if any ω-Cauchy sequence $\{p_n\}$ in C is convergent in C.

Mongkolkeha et al. [19] introduced the notion of contractive mapping in modular metric spaces as follows.

Definition 4 ([19])**.** Let ω be a metric modular on M and M_ω be a modular metric space induced by ω and $\Omega\colon M_\omega \to M_\omega$ be an arbitrary mapping Ω is called a ω-contraction if for each $p, q \in M_\omega$ and for all $\lambda > 0$ there exists $0 \leq k < 1$ such that

$$\omega_\lambda(\Omega p, \Omega q) \leq k\omega_\lambda(p,q). \tag{1}$$

Ege and Alaca[10] introduced the concept of modular b-metric space by introducing the scale factor in triangular inequality of usual metric space $b \geq 1$, which is a real number by generalizing the concept of modular metric space.

Definition 5 ([10])**.** Let M be a non-empty set and let $b \geq 1$ be a real number. A map $\nu_\lambda : (0,\infty) \times M_\nu \times M_\nu \to [0,\infty]$ is called a modular b-metric, if the following statements hold for all $p, q, r \in M_\nu$,

(i) $\nu_\lambda(p,q) = 0$ for all $\lambda > 0 \Leftrightarrow p = q$;
(ii) $\nu_\lambda(p,q) = \nu_\lambda(q,p)$ for all $\lambda > 0$;
(iii) $\nu_{\lambda+\mu}(p,q) \leq b[\nu_\lambda(p,r) + \nu_\mu(r,q)]$ for all $\lambda, \mu > 0$.

Then we say that (M_ν, ν_λ) is a modular b-metric space.
A modular set is defined as:

$$M_\nu = \{p \in M : \exists \lambda = \lambda(p) > 0 \text{ such that } \nu_\lambda(p,p_0) < \infty\}(p_0 \in X).$$

Definition 6 ([10])**.** Let (M_ν, ν_λ) be a modular b-metric space.

(i) A sequence $\{p_n\}_{n\in\mathbb{N}}$ in M_ν is called ν-convergent to $p \in M_\nu$ if $\nu_\lambda(p_n,p) \to 0$, as $n \to \infty$ for all $\lambda > 0$,
(ii) A sequence $\{p_n\}_{n\in\mathbb{N}} \subset M_\nu$ is said to be ν-Cauchy if and only if for all $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that for each $n, m \geq n(\epsilon)$ and $\lambda > 0$ we have $\nu_\lambda(p_n, p_m) < \epsilon$,
(iii) A modular b-metric space M_ν is ν-complete if each ν-Cauchy sequence in M_ν is ν-convergent and its limit is in M_ν.

Example 1 ([10])**.** Consider the space

$$l_x = \Big\{(p_n) \subset \mathbb{R} : \sum_{n=1}^{\infty} |p_n|^p < \infty\Big\}, \quad 0 < p < 1,$$

$\lambda \in (0,\infty)$ and $\nu_\lambda(p,q) = \dfrac{\varsigma(p,q)}{\lambda}$ such that

$$d(p,q) = \Big(\sum_{n=1}^{\infty} |p_n - q_n|^p\Big)^{\frac{1}{x}}, \quad p = p_n,\ q = q_n \in l_x.$$

It could be easily seen that (M_ν, ν_λ) is a modular b-metric space.

Rakotch [21] defined a new class of functions $\boldsymbol{\beta}$ as follows.

Definition 7 ([21])**.** Let (M,d) be a metric space. Denote by $\boldsymbol{\beta}$ the family of functions β satisfies the following conditions for all $p, q \in M$:

(i) $\beta(p,q) = \beta(d(p,q))$, i.e. β is dependent on the distance between p and q only;
(ii) $0 \leq \beta(d(p,q)) < 1$ for every $d(p,q) > 0$;
(iii) β is a monotonically decreasing function, where $0 \leq \beta < 1$.

Using the above concept, Geraghty [13] defined a class of functions $\boldsymbol{\beta}$ defined in a following way.

Definition 8 ([13])**.** Consider the family of all functions $\beta : \mathbb{R}^+ \to [0,1)$ as $\boldsymbol{\beta}$, where $\mathbb{R}^+ = \{t \in \mathbb{R} | t > 0\}$, such that t_n is monotonically decreasing in $\mathbb{R}^+$ which satisfy the condition

$$\lim_{n\to\infty} \beta\{t_n\} = 1 \text{ implies } \lim_{n\to\infty}\{t_n\} = 0$$

Theorem 1 ([13]). *Consider a Geraghty type contraction $\Omega : M \to M$ on a complete metric space satisfying*

$$d(\Omega(p), \Omega(q)) \leq \beta(d(p,q)).d(p,q)$$

where $\beta \in \boldsymbol{\beta}$. Then for any choice of initial point $p_0 \in M$, the iteration $p_n = \Omega p_{n-1}$ for $n \in \mathbb{N}_0$ converges to the unique fixed point p_0 of Ω in M.

Also Fulga and Proca [11] defined Geraghty type contraction as follows.

Definition 9 ([11]). Consider a metric space (M, d). Then a mapping $\Omega : M \to M$ is called a β_E-Geraghty type contraction on (M, d) if there exists $\beta \in \boldsymbol{\beta}$ such that:

$$d(\Omega p, \Omega q) \leq \beta(E(p,q)).E(p,q)$$

for all $p, q \in M$, where $E(p,q) = d(p,q) + |d(p, \Omega p) - d(q, \Omega q)|$.

In 2014, Popescu [20] introduced the notion of triangular α-orbital admissible as follows.

Definition 10 ([20]). For a non-empty set M, consider a self mapping $\Omega : M \to M$ and a function $\alpha : M \times M \to \mathbb{R}$. Then the mapping Ω is called a triangular α-orbital admissible, if for all $p, q \in M$

(Ω_1) $\alpha(p, \Omega p) \geq 1 \Rightarrow \alpha(\Omega p, \Omega^2 p) \geq 1$.
(Ω_2) $\alpha(p, q) \geq 1$ and $\alpha(q, \Omega q) \geq 1 \Rightarrow \alpha(p, \Omega q) \geq 1$.

Lemma 1 ([20]). *Consider a triangular α-orbital admissible mapping $\Omega : M \to M$. Assume that there exists p_0 such that $\alpha(p_0, \Omega p_0) \geq 1$. Define a sequence $\{p_n\}_{n \in \mathbb{N}}$ by $p_{n+1} = \Omega p_n$. Then, $\alpha(p_n, p_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $m > n$.*

Huang et al. [16] generalized the Geraghty function β for b-metric spaces as follows.

Definition 11 ([16]). Consider the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{b})$ denoted by $\boldsymbol{\beta}_b$, satisfying the following condition:

$$\lim_{n \to \infty} \sup \beta(t_n) = \frac{1}{b} \text{ implies that } \lim_{n \to \infty} t_n \to 0.$$

Aydi et al. [1] generalized the results of Fulga and Proca [12] by using the concept of Huang et al. [16] and introduced the notion of α-β_E-Geraghty contraction type mappings in b-metric spaces and proved some fixed point results as follows.

Definition 12 ([1]). Consider a b-metric space (M, d, b) and a function $\alpha : M \times M \to \mathbb{R}$. Then a mapping $\Omega : M \to M$ is called an α-β_E-Geraghty type contraction if there exists $\beta \in \boldsymbol{\beta}_b$ such that

$$\alpha(p,q) \geq 1 \Rightarrow d(\Omega p, \Omega q) \leq \beta(E(p,q)).E(p,q)$$

for all $p, q \in M$, where

$$E(p,q) = d(p,q) + |d(p, \Omega p) - d(q, \Omega q)|.$$

Lemma 2 ([1]). *Consider a b-metric space (M, d, b) and a convergent sequence $\{p_n\}_{n\in\mathbb{N}}$ in M with $\lim_{n\to\infty} p_n = p$. Then for all $p, q \in M$*

$$b^{-1}d(p, q) \leq \lim_{n\to\infty} \sup bd(p_n, q) \leq bd(p, q).$$

Theorem 2 ([1]). *Consider a complete b-metric space (M, d, b) and*

(i) *Ω is a triangular α-orbital admissible mapping;*
(ii) *Ω is an α-β_E -Geraghty type contractive mapping;*
(iii) *there exists $p_0 \in M$ such that $\alpha(p_0, \Omega p_0) \geq 1$;*
(iv) *Ω is continuous.*

Then Ω has a fixed point $r \in M$ and $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to r.

In [1] Aydi et al. replace the continuity of the mapping Ω with another condition and prove the following theorem.

Theorem 3 ([1]). *Consider a complete b-metric space (M, d, b) and a self mapping $\Omega : M \to M$. Suppose that the following*

(i) *Ω is a triangular α-orbital admissible mapping;*
(ii) *Ω is an α-β_E -Geraghty type contractive mapping;*
(iii) *there exists $p_0 \in M$ such that $\alpha(p_0, \Omega p_0) \geq 1$;*
(iv) *if $\{p_n\}_{n\in\mathbb{N}}$ is a sequence in M such that $\alpha(p_n, p_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $p_n \to p \in M$ as $n \to \infty$, then there exists a subsequence $\{p_{n_{(k)}}\}$ of $\{p_n\}_{n\in\mathbb{N}}$ such that $\alpha(p_{n_{(k)}}, p) \geq 1$ for all $k \in \mathbb{N}$. Then Ω has a fixed point $r \in M$ and $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to r.*

Wardowski [23] has defined a new class of function as follows.

Definition 13 ([23]). Let $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing on $\mathbb{R}^+$;
(F2) for every sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^+$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} F(t_n) = -\infty$;
(F3) there exists a number $k \in (0, 1)$ such that $\lim_{k\to 0} t^k F(t) = 0$ for all $t > 0$.

Definition 14 ([23]). For a metric space (M, d), a mapping $\Omega : M \to M$ is called a Wardowski's F-contraction if there exists $\tau > 0$ such that

$$d(p, q) > 1 \Rightarrow \tau + F(d(\Omega p, \Omega q)) \leq F(d(p, q)) \tag{2}$$

for all $p, q \in M$.

Example 2 ([23]). The following function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$:

(i) $F(t) = \ln t$, with $t > 0$;
(ii) $F(t) = \ln t + t$, with $t > 0$.

Cosentino et al. [8] generalized the Wardowski type contraction by introducing the following condition (**F4**).

Definition 15. (**F4**) Consider a real number $b \geq 1$, if there is a sequence $\{t_n\}_{n\in\mathbb{N}}$ of positive numbers such that $\tau + F(bt_n) \leq F(t_{n-1})$ for all $n \in \mathbb{N}$ and some $\tau > 0$, then

$$\tau + F(b^n t_n) \leq F(b^{n-1}t_{n-1}) \text{ for all } n \in \mathbb{N}. \tag{3}$$

Lukács and Kajántó [18] modified Wardowski's F-contraction and defined it in two steps on taking into account of Cosentino's condition (3).

Definition 16 ([18])**.** A function $F : \mathbb{R}^+ \to \mathbb{R}$ belongs to $\mathcal{F}$, if it satisfies the following conditions:

- (F_{l1}) F is strictly increasing;
- (F_{l2}) there exists $k \in (0, 1)$ such that $\lim_{k\to 0} t^k F(t) = 0$ for all $t > 0$.

Lukács and Kajántó [18] also defined a class $\mathcal{F}_{b,\tau}$ of F-contraction in b-metric spaces.

Definition 17 ([18])**.** Consider a b-metric space (M, d) with constant $b \geq 1$ and a self mapping $\Omega : M \to M$. If there exists $\tau > 0$ and $F \in \mathcal{F}_{b,\tau}$ such that for all $p, q \in M$, $d(\Omega p, \Omega q) > 0$, implies

$$\tau + F(bd(\Omega p, \Omega q)) \leq F(d(p, q)), \tag{F}$$

then Ω is called an F-contraction.

Lukács and Kajántó [18] proved the following results.

Theorem 4 ([18])**.** *If (M, d) is a complete b-metric space with constant $b \geq 1$ and $\Omega : M \to M$ is an F-contraction for some $F \in F_{b,\tau}$ then Ω has a unique fixed point p^*. Furthermore, for any $p_0 \in M$, the sequence $p_{n+1} = \Omega p_n$ is convergent and $\lim_{n\to\infty} p_n = p^*$.*

In 2017, Fulga and Proca [12] considered a new type of F-contraction as $\mathcal{F}_E$, a family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfies the following conditions:

- $(F_Y 1)$ F is strictly increasing, that is, for all $p, q \in \mathbb{R}^+$, if $p < q$ then $F(p) < F(q)$;
- $(F_Y 2)$ There exists $\tau > 0$ such that $\tau + \lim_{t\to t_0} \inf F(t) > \lim_{t\to t_0} \sup F(t)$, for every $t_0 > 0$.

Definition 18 ([12])**.** Consider a metric space (M, d). Then a self mapping $\Omega : M \to M$ is said to be a $\mathcal{F}_E$-contraction on (M, d), if there exists $F \in \mathcal{F}_E$ and $\tau > 0$ such that for all $p, q \in M$

$$d(p, q) > 1 \Rightarrow \tau + F(d(\Omega p, \Omega q)) \leq F(E(p, q)) \tag{4}$$

where

$$E(p, q) = d(p, q)+ \mid d(p, \Omega p) - d(q, \Omega q) \mid . \tag{5}$$

3. Main results

Before proving the main results some definitions have been stated as follows.

Definition 19. Let (M_ν, ν_λ, b) be a modular b-metric space. Let $\Omega : M_\nu \to M_\nu$ and $\alpha : M_\nu \times M_\nu \to [0, \infty)$ be two functions, then Ω is called a α-continuous mapping on (M_ν, ν_λ, b), if for a given $p \in M_\nu$, and sequence $\{p_n\}_{n\in\mathbb{N}}$ with

$$p_n \to p \text{ as } n \to \infty, \ \alpha(p_n, p_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}_0 \quad \Rightarrow \quad \Omega p_n \to \Omega p.$$

Now, α-β_Y-F-Geraghty type contraction is defined in the setting of modular b-metric space.

Definition 20. Let (M_ν, ν_λ, b) be a modular b-metric space such that $b \geq 1$ and $\alpha : M_\nu \times M_\nu \to \mathbb{R}$ be a function. A mapping $\Omega : M_\nu \to M_\nu$ is called an α-β_Y-F-Geraghty type contractive mapping, if there exists $\beta \in \boldsymbol{\beta}_b$ and $\tau > 0$ such that

$$\alpha(p, q) \geq 1 \implies \tau + F(b\nu_\lambda(\Omega p, \Omega q)) \leq \beta(Y(p, q))F(Y(p, q)), \tag{6}$$

where

$$Y(p, q) = \nu_\lambda(p, q) + \mid \nu_\lambda(p, \Omega p) - \nu_\lambda(q, \Omega q) \mid .$$

Remark 1. Since the functions belonging to $\boldsymbol{\beta}_b$ are strictly less than $\frac{1}{b}$, then $\beta(Y(p, q)) < \frac{1}{b}$, for all $p, q \in M_\nu$ with $\nu_\lambda(\Omega p, \Omega q) > 0$.

Lemma 3. *If $F : \mathbb{R}^+ \to \mathbb{R}$ is an increasing function and $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is a decreasing sequence such that $\lim_{n\to\infty} F(t_n) = -\infty$ then $\lim_{n\to\infty} t_n = 0$.*

Proof. Since $\{t_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below, it is also convergent. Let $\lim_{n\to\infty} t_n = a \geq 0$. If $a > 0$, $t_n \geq a$ and F is increasing, it follows that $F(a) \leq F(t_n)$, for all $n \geq 0$. Letting $n \to \infty$, we obtain $F(a) \leq \lim_{n\to\infty} F(t_n) = -\infty$, which is a contradiction, hence $a = 0$. □

Lemma 4. *Let (M_ν, ν_λ, b) be a modular b-metric space with constant $b \geq 1$, $p^*, q^* \in M$ and $\{p_n\}_{n\in\mathbb{N}}$ be a convergent sequence in M_ν with $\lim_{n\to\infty} p_n = p^*$. Then*

$$\begin{aligned} b^{-1}\nu_\lambda(p^*, q^*) &\leq \lim_{n\to\infty} \inf \nu_\lambda(p_n, q^*) \\ &\leq \lim_{n\to\infty} \sup \nu_\lambda(p_n, q^*) \\ &\leq b\nu_\lambda(p^*, q^*). \end{aligned}$$

Proof. If the triangle inequality of modular b-metric space is applied twice, for every $n \in \mathbb{N}$, then

$$\frac{1}{b}\varsigma(p^*, q^*) - \nu_\lambda(p_n, p^*) \leq \nu_\lambda(p_n, q^*) \leq b\nu_\lambda(p^*, q^*) + b\nu_\lambda(p_n, p^*).$$

If lim inf on the left-hand side inequality and lim sup on the right-hand side inequality is taken, then the desired property is proved. □

Theorem 5. *Let (M_ν, ν_λ, b) be a complete modular b-metric space. Suppose that the following conditions are satisfied:*

(ii) *Ω is a triangular α-orbital admissible mapping;*
(ii) *Ω is a α-β_Y-F-Geraghty type contractive mapping;*
(iii) *there exists $p_0 \in M_\nu$ such that $\alpha(p_0, \Omega p_0) \geq 1$;*
(iv) *Ω is continuous*

then, Ω has a fixed point $p^* \in M_\nu$ and $\{\Omega^n p_0\}_{n\in\mathbb{N}}$ converges to p^*.

Proof. By assumption (iii), there exists a point $p_0 \in M_\nu$ such that $\alpha(p_0, \Omega p_0) \geq 1$. Define a sequence $\{p_n\}_{n\in\mathbb{N}}$ in M_ν by $p_n = \Omega p_{n-1} = \Omega^n p_0$ for all $n \in \mathbb{N}$. Suppose that $p_n = p_{n+1} = \Omega p_n$ for some n, so the proof is completed.

Hence, suppose that $p_n \neq p_{n+1}$, then $\nu_\lambda(p_n, p_{n+1}) > 0$ for all $n \geq 0$. Now, denote $\nu_n = \nu_\lambda(p_{n-1}, p_n)$ for all $n \in \mathbb{N}_0$, $\alpha(p_0, p_1) = \alpha(p_0, \Omega p_0) \geq 1$. Since $\Omega : M_\nu \to M_\nu$ is α-orbital admissible, by induction it can be obtained

$$\alpha(p_n, p_{n+1}) = \alpha(\Omega p_{n-1}, \Omega\Omega p_{n-1}) = \alpha(\Omega^n p_0, \Omega^{n+1} p_0) \geq 1,$$

for all $n \in \mathbb{N}_0$.

Since Ω is triangular α-orbital admissible, then by Lemma 1 for $\alpha(p_n, p_{n+1}) \geq 1$ implies $\alpha(p_n, p_m) \geq 1$ for all $m, n \geq 0$ with $m > n$. Then, $\alpha(p_n, p_{n+1}) \geq 1$ implies

$$\begin{aligned} 0 < \tau + F(s.\nu_\lambda(p_n, p_{n+1})) &= \tau + F(s.\nu_\lambda(\Omega p_{n-1}, \Omega p_n)) \\ &\leq \beta(Y(p_{n-1}, p_n)) \cdot F(Y(p_{n-1}, p_n)), \end{aligned} \tag{7}$$

where

$$Y(p_{n-1}, p_n) = \nu_\lambda(p_{n-1}, p_n) + \big|\nu_\lambda(p_{n-1}, \Omega p_{n-1}) - \nu_\lambda(p_n, \Omega p_n)\big|. \tag{8}$$

Since $\nu_\lambda(p_{n-1}, p_n) = \nu_n$, then

$$Y(p_{n-1}, p_n) = \nu_n + |\nu_n - \nu_{n+1}|. \tag{9}$$

If $\nu_{n+1} > \nu_n$ then from (9)

$$Y(p_{n-1}, p_n) = \nu_{n+1}.$$

Therefore inequality (7) indicates that

$$F(b \cdot \nu_{n+1}) \leq \beta(\nu_{n+1}).F(\nu_{n+1}) - \tau \leq b^{-1}F(\nu_{n+1}) - \tau \leq b^{-1}F(b\nu_{n+1}) - \tau,$$

which is a contradiction. Thus, for all $n \in \mathbb{N}_0$, $\nu_{n+1} < \nu_n$. Therefore

$$F(b \cdot \nu_{n+1}) \leq F(\nu_n) - \tau$$

Hence, apply condition (3) to get

$$F(b^{n+1}\nu_{n+1}) \leq F(b^n \nu_n) - \tau, \quad \text{for all} \quad n \in \mathbb{N}. \tag{10}$$

Thus,

$$F(b^n\nu_n) \leq F(b^{n-1}\nu_{n-1}) - \tau \leq F(b^{n-2}\nu_{n-2}) - 2\tau \leq \cdots \leq F(\nu_0) - n\tau. \tag{11}$$

for all $n \in \mathbb{N}_0$. Since

$$\lim_{n\to\infty} F(\nu_0) - n\tau = -\infty.$$

Hence (11) implies
$$\lim_{n\to\infty} F(b^n\nu_n) = -\infty.$$
Thus, by condition (11) the sequence $(b^n\nu_n)_{n\in\mathbb{N}}$ is decreasing and the Lemma 3 can be applied to get $\lim_{n\to\infty} b^n\nu_n = 0$. According to $(F3)$, there exists $k \in (0,1)$ such that $\lim_{n\to\infty}(b^n\nu_n)^k \cdot F(b^n\nu_n) = 0$. Multiplying condition (11) by $(b^n\nu_n)^k$ results
$$0 \le n(b^n\nu_n)^k\tau + (b^n\nu_n)^k \cdot F(b^n\nu_n) \le (b^n\nu_n)^k \cdot F(\nu_0) \text{ for all } n \in \mathbb{N}.$$
If $n \to \infty$, then $\lim_{n\to\infty} n(b^n\nu_n)^k = 0$. Therefore, there exists $n_0 \in \mathbb{N}$ such that $n(b^n\nu_n)^k \le 1$ for all $n \ge n_1$. Thus, $\nu_n \le \dfrac{1}{b^n \cdot n^{\frac{1}{k}}}$ or
$$\nu_\lambda(p_{n-1}, p_n) \le \frac{1}{b^n \cdot n^{\frac{1}{k}}} \text{ for all } n \ge n_0.$$
Next, $\{p_n\}_{n\in\mathbb{N}}$ is proved as a Cauchy sequence. Suppose $m,n \in \mathbb{N}$ and $m > n$. Observe that, for $\frac{\lambda}{m-n} > 0$ there exists $n_i = n_{\frac{\lambda}{m-n}} \in \mathbb{N}$ such that for all $n \ge n_{\frac{\lambda}{m-n}}$. Then, for all $m > n$
$$\begin{aligned}\nu_\lambda(p_n, p_m) &\le \nu_{\frac{\lambda}{m-n}}(p_n, p_{n+1}) + \nu_{\frac{\lambda}{m-n}}(p_{n+1}, p_{n+2}) + \cdots + \nu_{\frac{\lambda}{m-n}}(p_{m-1}, p_m)\\ &\le \frac{1}{b^{n+1}\cdot(n+1)^{\frac{1}{k}}} + \frac{1}{b^{n+2}\cdot(n+2)^{\frac{1}{k}}} + \cdots + \frac{1}{b^m\cdot(m)^{\frac{1}{k}}}\\ &= \sum_{i=n+1}^{m} \frac{1}{b^i\cdot i^{\frac{1}{k}}} < \sum_{i=n+1}^{\infty} \frac{1}{b^i\cdot i^{\frac{1}{k}}}. \end{aligned} \tag{12}$$
Since the series $\sum_{i=n+1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, hence for all $n \ge n_0$ condition (12) implies
$$\nu_\lambda(p_n, p_m) \le \sum_{i=n+1}^{\infty} \frac{1}{b^i\cdot i^{\frac{1}{k}}} \to 0.$$
Thus, $\{p_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. So, the completeness of (M_ν, ν_λ, b) implies that there exists $p^* \in M_\nu$ such that $\lim_{n\to\infty} \nu_\lambda(p_n, p^*) = 0$ for $\alpha(p_n, p^*) \ge 1$, for all $n \in \mathbb{N}_0$. Here it is claimed that $p^* = \Omega p^*$. Since Ω is continuous, therefore $\lim_{n\to\infty} \nu_\lambda(\Omega p_n, \Omega p^*) = 0$, i.e., $\lim_{n\to\infty} \nu_\lambda(p_{n+1}, \Omega p^*) = 0$, which implies
$$\lim_{n\to\infty} \nu_\lambda(p^*, \Omega p^*) = 0.$$
Hence, $p^* = \Omega p^*$. □

Now, the uniqueness of such fixed point is to be proved. For this, the following additional condition is required:

(C) $\quad$ For all $p, q \in Fix(\Omega)$, we have $\alpha(p,q) \ge 1$,

where $Fix(\Omega)$ denotes the set of fixed point of Ω.

Theorem 6. *Addition of condition* (C) *to the hypothesis of Theorems 5 gives* p^* *as the unique fixed point of* Ω.

Proof. Suppose that there exists another fixed point $q^* \in M_\nu$, such that $p^* \neq q^*$, then for $\Omega q^* = q^*$. It is obvious that $\nu_\lambda(p^*, q^*) = \nu_\lambda(\Omega p^*, \Omega q^*) > 0$, also $\alpha(p^*, q^*) \geq 1$, it follows that

$$\begin{aligned} F(\nu_\lambda(\Omega p^*, \Omega q^*)) &\leq F(b\nu_\lambda((\Omega p^*, \Omega q^*)) \\ &\leq \beta(Y(p^*, q^*)).F(Y(p^*, q^*)) - \tau \\ &\leq b^{-1}.F(Y(p^*, q^*)) - \tau \\ &< b^{-1}.F(Y(p^*, q^*)) < F(Y(p^*, q^*)) \end{aligned} \tag{13}$$

where $Y(p^*, q^*) = \nu_\lambda(p^*, q^*) + \mid \nu_\lambda(p^*, \Omega p^*) - \nu_\lambda(q^*, \Omega q^*) \mid$, which implies

$$Y(p^*, q^*) = \nu_\lambda(p^*, q^*).$$

Therefore, from inequality (13)

$$F(\nu_\lambda(\Omega p^*, \Omega q^*)) \leq F(\nu_\lambda(p^*, q^*)) = F(\nu_\lambda(\Omega p^*, \Omega q^*)).$$

This is the contradiction. Hence $p^* = q^*$. □

Example 3. Let

$$M_\nu = \left[0, \tfrac{7}{10}\right] \bigcup \{1\}, \quad \nu_\lambda(p, q) = \frac{|p-q|^2}{\lambda}, \quad \Omega : M_\nu \to M_\nu,$$

such that

$$\Omega(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 \leq x \leq \frac{7}{10}; \\ \frac{1}{4}, & \text{if } x = 1; \end{cases}$$

and choosing $F(t) = \ln t, t \in (0, \infty)$ and $\tau = \ln 2$ and $\beta(t) = \frac{1}{4}$.

For $p = x \in [0, \frac{7}{10}]$ and $q = 1$ we have

$$\begin{aligned} \tau + F(2\nu_\lambda(\Omega x, \Omega 1)) &= \ln 2 + F(2\nu_\lambda(\tfrac{x}{2}, \tfrac{1}{4})) \\ &= \ln 2 + F\left(2\frac{|2x-1|^2}{16\lambda}\right) \\ &= \ln 2 + F\left(\frac{|2x-1|^2}{8\lambda}\right) \\ &= \ln 2 + \ln\left(\frac{|2x-1|^2}{8\lambda}\right) \\ &= \ln\left(\frac{\mid 2x-1 \mid^2}{4\lambda}\right), \end{aligned} \tag{14}$$

$$
\begin{aligned}
Y(x,1) &= \nu_\lambda(x,1) + \left|\nu_\lambda(x,\Omega x) - \nu_\lambda(1,\Omega 1)\right| \\
&= \frac{|x-1|^2}{\lambda} + \left| \frac{\left|x-\frac{x}{2}\right|^2}{\lambda} - \frac{\left|1-\frac{1}{4}\right|^2}{\lambda} \right| \\
&= \frac{|x-1|^2}{\lambda} + \left| \frac{x^2}{4\lambda} - \frac{9}{16\lambda} \right| \\
&= \frac{16|x-1|^2 + |4x^2-9|}{16\lambda}. \qquad (15)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\beta(Y(p,q)) \cdot F(Y(p,q)) &= \frac{1}{4} \cdot F\left(\frac{16|x-1|^2 + |4x^2-9|}{16\lambda}\right) \\
&= \frac{1}{4} \cdot \ln\left(\frac{16|x-1|^2 + |4x^2-9|}{16\lambda}\right) \\
&= \ln\left(\frac{16|x-1|^2 + |4x^2-9|}{16\lambda}\right)^{\frac{1}{4}}. \qquad (16)
\end{aligned}
$$

So, from inequalities (14) and (16), it is clear that all the conditions of Theorem 6 holds for $\lambda > 0$. Here $p = 0$ is the unique fixed point.

Letting $\alpha(p,q) = 1$ for all $p, q \in \mathbb{R}$ in Theorem 5, states the following corollaries.

Corollary 1. *Consider a complete modular b-metric space* (M_ν, ν_λ, b) *and a continuous mapping* $\Omega : M_\nu \to M_\nu$. *Suppose there exists* $\beta \in \boldsymbol{\beta}_{b,\tau}$ *such that*

$$\tau + F(b\nu_\lambda(\Omega p, \Omega q)) \le \beta(Y(p,q)) \cdot F(Y(p,q)), \qquad (17)$$

for all $p, q \in M_\nu$, *where*

$$Y(p,q) = \nu_\lambda(p,q) + \left|\nu_\lambda(p,\Omega p) - \nu_\lambda(q,\Omega q)\right|.$$

Then Ω *has a fixed point* $p^* \in M_\nu$ *and* $\{\Omega^n p\}_{n\in\mathbb{N}}$ *converges to* p^* *for any* $p \in M_\nu$.

The following two consequences can also be stated.

Corollary 2. *Consider a complete modular b-metric space* (M_ν, ν_λ, b) *be and a continuous mapping* $\Omega : M_\nu \to M_\nu$ *such that*

$$\tau + F(b\nu_\lambda(\Omega p, \Omega q)) \le \frac{F(Y(p,q))}{b + Y(p,q)}. \qquad (18)$$

for all $p, q \in M_\nu$, *where*

$$Y(p,q) = \nu_\lambda(p,q) + \left|\nu_\lambda(p,\Omega p) - \nu_\lambda(q,\Omega q)\right|.$$

Then Ω *has a fixed point* $p^* \in M_\nu$ *and* $\{\Omega^n p\}$ *converges to* p^* *for any* $p \in M_\nu$.

Proof. Take

$$\beta(t) = \begin{cases} \frac{1}{b+t}, & \text{if } t > 0; \\ \frac{1}{b+1}, & \text{if } t = 0; \end{cases}$$

where $\beta \in \boldsymbol{\beta}_{b,\tau}$ in Corollary 1 for $p \neq q$ and for $p = q$, the proof can be done. □

Corollary 3. *Let (M_ν, ν_λ, b) be a metric space. A continuous map $\Omega : M_\nu \to M_\nu$ is called an F-contraction on (M_ν, ν_λ, b) if there exists $F \in \mathcal{F}$, where $\tau > 0$ such that for all $p, q \in M_\nu$*

$$\nu_\lambda(\Omega p, \Omega q) > 0 \ \Rightarrow \ \tau + F(\nu_\lambda(\Omega p, \Omega q)) \leq F(Y(p,q)). \tag{19}$$

for all $p, q \in M_\nu$, where

$$Y(p,q) = \nu_\lambda(p,q) + \big|\nu_\lambda(p, \Omega p) - \nu_\lambda(q, \Omega q)\big|.$$

Then Ω has a unique fixed point $p^* \in M_\nu$ and $\{\Omega^n p\}_{n\in\mathbb{N}}$ converges to p^* for any $p \in M_\nu$.

Proof. Since for $b = 1$, $\beta(t) < 1$, where $t > 0$. So, by applying this in Corollary 1, the proof is done. □

4. Conclusion

From the above discussion it is clear that, the proved results generalize and extend the results of Aydi et al. (see [1]), Lukács and Kajántó (see [18]) and the results of Fulga and Proca (see [12]), also extend and unify several results from the literature. These results can also be generalized by replacing the modular b-metric space with some other abstract spaces such as generalized modular metric space, non-Archimedean modular metric space etc.

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